JOURNAL OF MATERIALS SCIENCE 24 (1989) 3865–3872

Stress disturbance due to broken fibres in metal matrix composites with non-uniform fibre spacing

SHOJIRO OCHIAI, KOZO OSAMURA
Department of Metallurgy, Kyoto University, Sakyo-ku, Kyoto 606, Japan

Premature fracture of weaker fibres causes stress disturbances in composites. These disturbances are affected by non-uniformity of fibre spacing. In order to evaluate quantitatively how the disturbances in metal matrix composites are affected by the extent of non-uniformity of fibre spacing, a method of calculation is presented on the basis of two-dimensional shear lag analysis. Static tensile stress concentrations in the intact fibres to broken fibres, tensile stress distribution along the fibre axis in the broken and intact fibres and shear stresses between broken and intact fibres were calculated by the method presented, using some examples. It is shown quantitatively that the spacing between broken and intact fibres and that between intact and next fibres has a significant influence on tensile stress concentrations in intact fibres and also on the shear stresses between broken and intact fibres: the narrower the former spacing and the wider the latter spacing, the higher become both tensile and shear stress concentrations. This tendency is enhanced when the number of broken fibres is large and when the strain hardening of the matrix is high.

1. Introduction

The premature fracture of weaker fibres in fibre-reinforced composites causes stress concentrations along the fibre axis in the fibres adjacent to broken fibres, stress reduction near the broken ends in broken fibres and shear stress concentration between broken and intact fibres. These stress disturbances have been quantitatively analysed for elastic fibre–elastic matrix composites [1, 2], and for elastic fibre–metal matrix composites [3, 4]. In these analyses, it has been assumed that fibre spacing is uniform. This assumption is, however, not necessarily practical, because most composites have, more or less, non-uniformity of fibre spacing.

The aim of the present paper is to present a method to calculate the static stress disturbances for elastic fibre–strain hardenable metal matrix composites whose fibre spacing is not uniform. In the following paper [5], employing the method of calculation presented here, how the non-uniformity of fibre spacing affects the tensile strength of metal matrix composites will be studied by means of a Monte-Carlo simulation technique.

In the present method of calculation, shear lag analysis for a two-dimensional array of fibres [1–4] was modified under an approximation that only intact fibres are subjected to the stress concentrations resulting from the broken fibres and the fibres outside the intact fibres undergo uniform deformation in tension along the fibre axis. This approximation has been known as a useful tool for studying nonelastic effects of the matrix [3].

2. The method of calculation

2.1. Shear stress–shear strain curve of the matrix

The shear stress (τ_m)–shear strain (γ) curve of the matrix was approximated as shown in Fig. 1. In the stage of elastic deformation ($\gamma < \gamma_y$ where γ_y is the yield strain in shear), τ_m was given by

$$\tau_m = G_m \gamma \qquad (1)$$

where G_m is the shear modulus of the matrix, and in the stage of plastic shear deformation ($\gamma > \gamma_y$), it was given by

$$\tau_m = \beta G_m \gamma + (1 - \beta)\tau_y \qquad (2)$$

where τ_y is the shear yield stress given by $G_m \gamma_y$ and β is the slope of shear stress–shear strain curve in plastic deformation, normalized with respect to G_m. "$\beta = 0$", "$0 < \beta < 1$" and "$\beta = 1$" mean that the matrix exhibits no strain hardening after yielding in shear, it deforms with strain hardening coefficient βG_m, and it deforms elastically, respectively.

2.2. Model structure of the composites

Fig. 2 shows the two-dimensional model employed in the present calculation. This model is similar to that of Zweben [3], but it is different in that the strain hardening of the matrix after yield in shear and non-uniformity of fibre spacing are taken into consideration.

The model consists of a central core of n broken fibres shown by "2" in Fig. 2, two intact fibres shown by "1" and "3", and the next two fibres shown by "0"

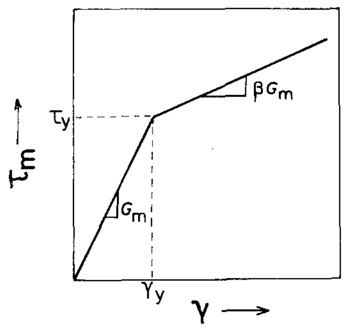

Figure 1 Schematic representation of the shear stress–shear strain curve of the metal matrix employed in the present calculation.

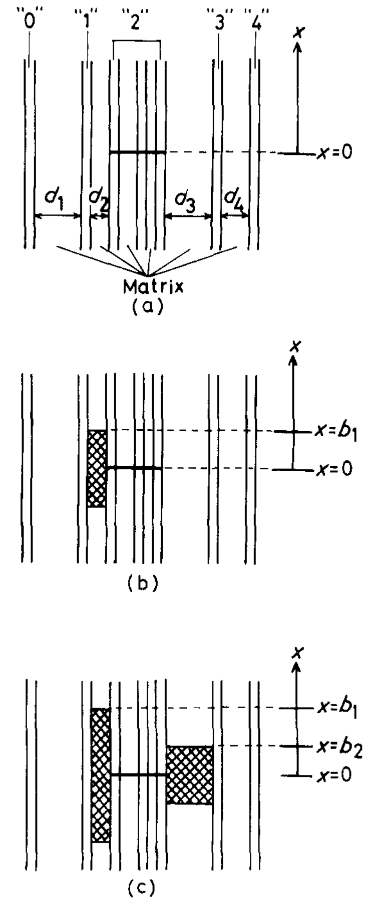

Figure 2 Schematic representation of the model composite employed in the present calculation. "2" indicates n broken fibres, "1" and "2" the intact fibres and "0" and "4" the next fibres. (a), (b) and (c) correspond to Stages I, II and III, respectively. The hatched regions show the regions of matrix yielded in shear.

and "4". The fibre spacings between "0" and "1", "1" and "2", "2" and "3", "3" and "4" fibres are given by d_1, d_2, d_3 and d_4, respectively.

In the present calculation, three assumptions were made for simplicity. (i) The shear strain in the core of cut fibres could be neglected. This assumption makes the mathematical treatment easy in that the displacement of the fibres in the core can be given as a function only of x, the axial coordinate parallel to fibres where x is taken to be zero at the cross-section where the fibres "2" are cut as shown in Fig. 2a. (ii) Only the intact fibres are subjected to stress concentration, as stated above. This assumption leads to the simple expression that the fibres "0" and "4" undergo uniform deformation in the x direction. (iii) The bonding strength between fibres and matrix is high enough to suppress debonding.

2.3. Deformation stages

In the present model, the following three stages arise with increasing stress level, as shown in Fig. 2.

Stage I: when the stress level is low, the shear stresses between "1" and "2" ($\tau_{1\text{-}2}$) and "2" and "3" fibres ($\tau_{2\text{-}3}$) at any x are lower than τ_y.

Stage II: the $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ have maxima at $x = 0$ and decrease with increasing x, as shown later. With increasing applied stress level, one of the shear stresses of $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ exceeds τ_y at $x = 0$, at which stress level the matrix between "1" and "2" fibres begins yielding in shear if $\tau_y = \tau_{1\text{-}2} > \tau_{2\text{-}3}$ at $x = 0$ or the matrix between "2" and "3" fibres begins yielding if $\tau_y = \tau_{2\text{-}3} > \tau_{1\text{-}2}$. With further increasing stress level, the region of yielding of matrix grows. In Fig. 2b, the case where the $\tau_{1\text{-}2}$ exceeds τ_y for the region of $0 \leqslant x \leqslant b_1$ but not $\tau_{2\text{-}3}$ at any x, is illustrated as an example. In this stage, two regions exist; Region A where both $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ are lower than τ_y and Region B where one of $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ exceeds τ_y at $0 \leqslant x \leqslant b_1$. Region A covers the region of $x \geqslant b_1$ and Region B the region of $0 \leqslant x \leqslant b_1$. Taking the case of Fig. 2b, $\tau_{1\text{-}2}$ is equal to τ_y at $x = b_1$.

Stage III: when stress levels become high, both $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ exceed τ_y at least at $x = 0$. Fig. 2c shows an illustration of this stage for the case of $\tau_{1\text{-}2} > \tau_{2\text{-}3}$. In this stage, a new region (described as Region C) arises, where both $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ exceed τ_y, in addition to Regions A and B. Noting the length of Region C as b_2 and that of Region B as $b_1 - b_2$, Regions A, B and C cover the regions of $b_1 \leqslant x$, $b_2 \leqslant x \leqslant b_1$ and $0 \leqslant x \leqslant b_2$, respectively. In this stage, $\tau_{1\text{-}2}$ and $\tau_{2\text{-}3}$ are equal to τ_y at $x = b_1$ and b_2, respectively.

2.4. Equations for stress equilibrium

Representing the displacements of "0" to "4" fibres by U_0 to U_4, respectively, U_0 and U_4 are given by $\sigma_f x/E_f$ from the assumption (ii) above, where σ_f is the stress of fibre at $x = \infty$ and E_f is the Young's modulus of the fibre. Whether $\tau_{1\text{-}2}$ or $\tau_{2\text{-}3}$ becomes τ_y first depends on the values of d_1 to d_4. In the following procedure, we show equations for the case where $\tau_{1\text{-}2}$ becomes τ_y prior to $\tau_{2\text{-}3}$. Also, equations for the case where $\tau_{2\text{-}3}$ becomes τ_y prior to $\tau_{1\text{-}2}$ can be derived similarly. For Regions A, B and C, the equilibrium equations for U_1 to U_3 are given as follows, by taking Equations 1 and 2 into consideration. For Region A

$$E_f A_f (\mathrm{d}^2 U_1/\mathrm{d}x^2) = -G_m h(U_2 - U_1)/d_2 + G_m h(U_1 - \sigma_f x/E_f)/d_1 \quad (3)$$

$$nE_f A_f (\mathrm{d}^2 U_2/\mathrm{d}x^2) = G_m h(U_2 - U_1)/d_2 + G_m h(U_2 - U_3)/d_3 \quad (4)$$

$$E_f A_f (\mathrm{d}^2 U_3/\mathrm{d}x^2) = -G_m h(U_2 - U_3)/d_3 + G_m h(U_3 - \sigma_f x/E_f) d_4 \quad (5)$$

For Region B

$$E_f A_f (d^2 U_1/dx^2) = -[\beta G_m (U_2 - U_1)/d_2 + (1-\beta)\tau_y]h + G_m h(U_1 - \sigma_f x/E_f)d_1 \quad (6)$$

$$nE_f A_f (d^2 U_2/dx^2) = [\beta G_m (U_2 - U_1)/d_2 + (1-\beta)\tau_y]h + G_m h(U_2 - U_3)/d_3 \quad (7)$$

$$E_f A_f (d^2 U_3/dx^2) = -G_m h(U_2 - U_3)/d_3 + G_m h(U_3 - \sigma_f x/E_f)d_4 \quad (8)$$

For Region C

$$E_f A_f (d^2 U_1/dx^2) = -[\beta G_m (U_2 - U_1)/d_2 + (1-\beta)\tau_y]h + G_m h(U_1 - \sigma_f x/E_f)/d_1 \quad (9)$$

$$nE_f A_f (d^2 U_2/dx^2) = [\beta G_m (U_2 - U_1)/d_2 + (1-\beta)\tau_y]h + [\beta G_m (U_2 - U_3)/d_3 + (1-\beta)\tau_y]h \quad (10)$$

$$E_f A_f (d^2 U_3/dx^2) = -[\beta G_m (U_2 - U_3)/d_3 + (1-\beta)\tau_y]h + G_m h(U_3 - \sigma_f x/E_f)/d_4 \quad (11)$$

where A_f is the cross-sectional area of fibre and h the thickness of the model composite.

2.5. Non-dimensionalization

In order to obtain a convenient form for the problem, non-dimensionalization was carried out by modifying the method of Hedgepath who first introduced the idea of non-dimensionalization to solve this kind of equation for the model composite where fibre spacing is uniform [1].

The average fibre spacing, d_{av}, is given by $(d_1 + d_2 + d_3 + d_4)/4$. Using d_{av}, E_f, A_f, G_m, h and σ_f, non-dimensionalization of U_1 to U_3, τ_y, x, b_1 and b_2 was carried out by letting

$$U_{1(2,3)} = \sigma_f (A_f d_{av}/E_f G_m h)^{1/2} u_{1(2,3)} \quad (12)$$

$$\tau_y = \sigma_f (G_m A_f/E_f d_{av} h)^{1/2} \bar{\tau}_y \quad (13)$$

$$x(b_1, b_2) = (E_f A_f d_{av}/G_m h)^{1/2} \xi(\bar{b}_1, \bar{b}_2) \quad (14)$$

where $u_{1(2,3)}$, $\bar{\tau}_y$ and $\xi(\bar{b}_1, \bar{b}_2)$ are non-dimensionalized forms of $U_{1(2,3)}$, τ_y and $x(b_1, b_2)$, respectively.

Defining

$$d_1 = f_1 d_{av}, \quad d_2 = f_2 d_{av}, \quad d_3 = f_3 d_{av} \quad \text{and} \quad d_4 = f_4 d_{av} \quad (15)$$

we can regard f_1 to f_4 as measures of non-uniformity. In this definition, $f_1 + f_2 + f_3 + f_4$ is always equal to 4. For convenience, we again define

$$a_1 = 1/f_1, \quad a_2 = 1/f_2, \quad a_3 = 1/f_3 \quad \text{and} \quad a_4 = 1/f_4 \quad (16)$$

Combining Equations 3 to 16, we have convenient forms of non-dimensionalized equations for Regions A to C, as follows. For Region A

$$d^2u_1/d\xi^2 = -a_2(u_2 - u_1) + a_1(u_1 - \xi) \quad (17)$$

$$n(d^2u_2/d\xi^2) = a_2(u_2 - u_1) + a_3(u_2 - u_3) \quad (18)$$

$$d^2u_3/d\xi^2 = -a_3(u_2 - u_3) + a_4(u_3 - \xi) \quad (19)$$

For Region B

$$d^2u_1/d\xi^2 = -[\beta a_2(u_2 - u_1) + (1-\beta)\bar{\tau}_y] + a_1(u_1 - \xi) \quad (20)$$

$$n(d^2u_2/d\xi^2) = \beta a_2(u_2 - u_1) + (1-\beta)\bar{\tau}_y + a_3(u_2 - u_3) \quad (21)$$

$$d^2u_3/d\xi^2 = -a_3(u_2 - u_3) + a_4(u_3 - \xi) \quad (22)$$

For Region C

$$d^2u_1/d\xi^2 = -[\beta a_2(u_2 - u_1) + (1-\beta)\bar{\tau}_y] + a_1(u_1 - \xi) \quad (23)$$

$$n(d^2u_2/d\xi^2) = [\beta a_2(u_2 - u_1) + (1-\beta)\bar{\tau}_y] + [\beta a_3(u_2 - u_3) + (1-\beta)\bar{\tau}_y] \quad (24)$$

$$d^2u_3/d\xi^2 = -[\beta a_3(u_2 - u_3) + (1-\beta)\bar{\tau}_y] + a_4(u_3 - \xi) \quad (25)$$

2.6. Solutions

2.6.1. Region A

The solutions of Equations 17 and 19 for Region A are

$$u_1^A = \xi + \sum_{i=1}^{6} A_i \exp(k_i\xi) \quad (26)$$

$$u_2^A = \xi + \sum_{i=1}^{6} A_i(-k_i^2/a_2 + 1 + a_1/a_2) \times \exp(k_i\xi) \quad (27)$$

$$u_3^A = \xi + \sum_{i=1}^{6} A_i[nk_i^4/a_2a_3 - [a_2 + a_3 + n(a_1 + a_2)]k_i^2 + (a_2a_3 + a_1a_2 + a_1a_3)/a_2a_3] \times \exp(k_i\xi) \quad (28)$$

where A_1 to A_6 are integral constants and the superscript A to u_1 to u_3 means Region A. k_1 to k_6 are given as the solutions of

$$(n/a_2a_3)k^6 - \{[a_2 + a_3 + n(a_1 + a_2 + a_3 + a_4)]/a_2a_3\}k^4 + \{[2a_2a_3 + (a_1 + a_4)(a_2 + a_3) + n(a_1 + a_2)(a_3 + a_4)]/a_2a_3\}k^2 - [a_2a_3(a_1 + a_4) + a_1a_4(a_2 + a_3)]/a_2a_3 = 0 \quad (29)$$

2.6.2. Region B

The solutions of Equations 20 to 22 for Region B and Equations 23 to 25 for Region C have different forms depending on the value of β.

2.6.2.1. $\beta \neq 0$. In the case of $\beta \neq 0$, the solutions of Equations 20 to 22 are

$$u_1^B = \xi + \{a_3a_4(1-\beta)/[\beta(a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4) + a_1a_3a_4]\}\bar{\tau}_y + \sum_{i=1}^{6} B_i \exp(t_i\xi) \quad (30)$$

$$u_2^B = \xi - \{a_1(a_3 + a_4)(1-\beta)/[\beta(a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4) + a_1a_3a_4]\}\bar{\tau}_y + \sum_{i=1}^{6} B_i[-t_i^2/(\beta a_2) + 1 + a_1/(\beta a_2)] \exp(t_i\xi) \quad (31)$$

$$u_3^{\mathrm{B}} = \xi - \{a_1a_3(1-\beta)/[\beta(a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4) + a_1a_3a_4]\}\bar{\tau}_y + \sum_{i=1}^{6} B_i\{nt_i^4/(\beta a_2a_3) - [\beta a_2 + a_3 + n(a_1+\beta a_2)]t_i^2/(\beta a_2a_3) + [\beta a_2(a_1+a_3) + a_1a_3]/(\beta a_2a_3)\}\exp(t_i\xi) \tag{32}$$

where B_1 to B_6 are integral constants and t_1 to t_6 are solutions of

$$(n/\beta a_2a_3)t^6 - \{[\beta a_2 + a_3 + n(a_1+\beta a_2+a_3+a_4)]/\beta a_2a_3\}t^4 + \{[2\beta a_2a_3 + n(a_1+\beta a_2)(a_3+a_4) + (\beta a_2+a_3)(a_1+a_4)]/\beta a_2a_3\}t^2 - \{[\beta(a_1a_2a_3 + a_1a_2a_4 + a_2a_3a_4) + a_1a_3a_4]/\beta a_2a_3\} = 0 \tag{33}$$

2.6.2.2. $\beta = 0$. In the case of $\beta = 0$, the solutions of Equations 20 to 22 are

$$u_1^{\mathrm{B}} = \xi + \bar{\tau}_y/a_1 + B_1\exp(-a_1^{1/2}\xi) + B_2\exp(a_1^{1/2}\xi) \tag{34}$$

$$u_2^{\mathrm{B}} = \xi - (1/a_3 + 1/a_4)\bar{\tau}_y + \sum_{i=3}^{6} B_i(-t_i^2/a_3 + 1 + a_4/a_3)\exp(t_i\xi) \tag{35}$$

$$u_3^{\mathrm{B}} = \xi - \bar{\tau}_y/a_4 + \sum_{i=3}^{6} B_i\exp(t_i\xi) \tag{36}$$

where t_3 to t_6 are solutions of

$$nt^4/a_3 - (n + 1 + na_4/a_3)t^2 + a_4 = 0 \tag{37}$$

2.6.3. Region C

2.6.3.1. $\beta \neq 0$. In the case of $\beta \neq 0$, the solutions of Equations 23 to 25 for Region C are

$$u_1^{\mathrm{C}} = \xi - \{a_4(a_2-a_3)(1-\beta)/[\beta(a_1a_2a_3 + a_2a_3a_4) + a_1a_2a_4 + a_1a_3a_4]\}\bar{\tau}_y + \sum_{i=1}^{6} C_i\exp(s_i\xi) \tag{38}$$

$$u_2^{\mathrm{C}} = \xi - \{[\beta(a_1a_3 + a_2a_4) + 2a_1a_4]/[\beta(a_1a_2a_3 + a_2a_3a_4) + a_1a_2a_4 + a_1a_3a_4]\}[(1-\beta)/\beta]\bar{\tau}_y + \sum_{i=1}^{6} C_i(-s_i^2/\beta a_2 + 1 + a_1/\beta a_2)\times\exp(s_i\xi) \tag{39}$$

$$u_3^{\mathrm{C}} = \xi + \{a_1(a_2-a_3)(1-\beta)/[\beta(a_1a_2a_3 + a_2a_3a_4) + a_1a_2a_4 + a_1a_3a_4]\}\bar{\tau}_y + \sum_{i=1}^{6} C_i\{ns_i^4/(\beta^2a_2a_3) - [\beta a_2(1+n) + na_1 + \beta a_3]s_i^2/(\beta^2a_2a_3) + (a_1a_2 + \beta a_2a_3 + a_1a_3)/(\beta a_2a_3)\}\exp(s_i\xi) \tag{40}$$

where C_1 to C_6 are integral constants and s_1 to s_6 are solutions of

$$[n/(\beta^2a_2a_3)]s^6 - \{[\beta(a_2+a_3) + n(a_1+\beta a_2+\beta a_3+a_4)]/(\beta^2a_2a_3)\}s^4 + \{[2\beta^2a_2a_3 + \beta(a_1+a_4)(a_2+a_3) + n(a_1+\beta a_2)\times(\beta a_3+a_4)]/(\beta^2a_2a_3)\}s^2 - [\beta(a_1a_2a_3 + a_2a_3a_4) + a_1a_2a_4 + a_1a_3a_4]/(\beta a_2a_3) = 0 \tag{41}$$

2.6.3.2. $\beta = 0$. In the case of $\beta = 0$, the solutions of Equations 23 to 25 are

$$u_1^{\mathrm{C}} = \xi + \bar{\tau}_y/a_1 + C_1\exp(-a_1^{1/2}\xi) + C_2\exp(a_1^{1/2}\xi) \tag{42}$$

$$u_2^{\mathrm{C}} = \bar{\tau}_y\xi^2/n + C_3\xi + C_4 \tag{43}$$

$$u_3^{\mathrm{C}} = \xi + \bar{\tau}_y/a_4 + C_5\exp(-a_4^{1/2}\xi) + C_6\exp(a_4^{1/2}\xi) \tag{44}$$

2.7. Boundary conditions

The stress concentration factors at $\xi = \xi$ in "1", "2" and "3" fibres, $K_1(\xi)$, $K_2(\xi)$ and $K_3(\xi)$, respectively, are given by

$$K_1(\xi) = \mathrm{d}u_1(\xi)/\mathrm{d}\xi,\quad K_2(\xi) = \mathrm{d}u_2(\xi)/\mathrm{d}\xi,\quad K_3(\xi) = \mathrm{d}u_3(\xi)/\mathrm{d}\xi \tag{45}$$

The positive non-dimensional shear stresses between "1" and "2" and "2" and "3" fibres, $\bar{\tau}_{1\text{-}2}$ and $\bar{\tau}_{2\text{-}3}$, respectively, in Region A are given by

$$\bar{\tau}_{1\text{-}2}(\xi) = a_2[u_2(\xi) - u_1(\xi)]$$
$$\bar{\tau}_{2\text{-}3}(\xi) = a_3[u_2(\xi) - u_3(\xi)] \tag{46}$$

In Region B, $\bar{\tau}_{1\text{-}2}(\xi)$ and $\bar{\tau}_{2\text{-}3}(\xi)$ are given by

$$\bar{\tau}_{1\text{-}2}(\xi) = \beta a_2[u_2(\xi) - u_1(\xi)] + (1-\beta)\bar{\tau}_y$$
$$\bar{\tau}_{2\text{-}3}(\xi) = a_3[u_2(\xi) - u_3(\xi)] \tag{47}$$

In Region C, they are given by

$$\left.\begin{aligned}\bar{\tau}_{1\text{-}2}(\xi) &= \beta a_2[u_2(\xi) - u_1(\xi)] + (1-\beta)\bar{\tau}_y\\ \bar{\tau}_{2\text{-}3}(\xi) &= \beta a_3[u_2(\xi) - u_3(\xi)] + (1-\beta)\bar{\tau}_y\end{aligned}\right. \tag{48}$$

The solutions of u_1 to u_3 in Regions B and C have different forms according to the value of β, as shown already, but the boundary conditions are common for any value of β. The boundary conditions are given for each stage as follows.

2.7.1. Stage I where only Region A exists

1. At $\xi = 0$, the displacements of fibres "1" and "3" are zero; $u_1^{\mathrm{A}}(0) = 0$, $u_3^{\mathrm{A}}(0) = 0$, and the stress of fibres "2" is zero; $K_2^{\mathrm{A}}(0) = 0$.

2. At $\xi = \infty$, stress concentration factors for all fibres are unity; $K_1^{\mathrm{A}}(\infty) = 1$, $K_2^{\mathrm{A}}(\infty) = 1$, $K_3^{\mathrm{A}}(\infty) = 1$.

2.7.2. Stage II where Regions A($\bar{b}_1 \leqslant \xi$) and B($0 \leqslant \xi \leqslant \bar{b}_1$) exist

1. At $\xi = 0$, the displacement of fibres "1" and "3" is zero; $u_1^{\mathrm{B}}(0) = 0$, $u_3^{\mathrm{B}}(0) = 0$, and the stress of fibres "2" is zero; $K_2^{\mathrm{B}}(0) = 0$.

2. At $\xi = \bar{b}_1$, displacements of fibres should be continuous; $u_1^{\mathrm{A}}(\bar{b}_1) = u_1^{\mathrm{B}}(\bar{b}_1)$, $u_2^{\mathrm{A}}(\bar{b}_1) = u_2^{\mathrm{B}}(\bar{b}_1)$,

$u_3^A(\bar{b}_1) = u_3^B(\bar{b}_1)$, the stresses of fibres should be continuous; $K_1^A(\bar{b}_1) = K_1^B(\bar{b}_1)$, $K_2^A(\bar{b}_1) = K_2^B(\bar{b}_1)$, $K_3^A(\bar{b}_1) = K_3^B(\bar{b}_1)$, and the shear stress between fibres "1" and "2" is equal to τ_y; $\bar{\tau}_{1\text{-}2}(\bar{b}_1) = \bar{\tau}_y$.

3. At $\xi = \infty$, stress concentration factors for all fibres are unity; $K_1^A(\infty) = 1, K_2^A(\infty) = 1, K_3^A(\infty) = 1$.

2.7.3. Stage III where Regions $A(\bar{b}_1 \leqslant \xi)$, $B(\bar{b}_2 \leqslant \xi \leqslant \bar{b}_1)$ and $C(0 \leqslant \xi \leqslant \bar{b}_2)$ exist

1. At $\xi = 0$, the displacement of fibres "1" and "3" is zero; $u_1^C(0) = 0$, $u_3^C(0) = 0$, and the stress of fibres "2" is zero; $K_2^C(0) = 0$.

2. At $\xi = \bar{b}_2$, the displacements of fibres should be continuous; $u_1^B(\bar{b}_2) = u_1^C(\bar{b}_2)$, $u_2^B(\bar{b}_2) = u_2^C(\bar{b}_2)$, $u_3^B(\bar{b}_2) = u_3^C(\bar{b}_2)$, the stresses of fibres should be continuous; $K_1^B(\bar{b}_2) = K_1^C(\bar{b}_2)$, $K_2^B(\bar{b}_2) = K_2^C(\bar{b}_2)$, $K_3^B(\bar{b}_2) = K_3^C(\bar{b}_2)$, and the shear between fibres "2" and "3" is equal to τ_y; $\bar{\tau}_{2\text{-}3}(\bar{b}_2) = \bar{\tau}_y$.

3. At $\xi = \bar{b}_1$, as similarly as at $\xi = \bar{b}_2$, the displacements and stresses of fibres should be continuous; $u_1^A(\bar{b}_1) = u_1^B(\bar{b}_1)$, $u_2^A(\bar{b}_1) = u_2^B(\bar{b}_1)$, $u_3^A(\bar{b}_1) = u_3^B(\bar{b}_1)$, $K_1^A(\bar{b}_1) = K_1^B(\bar{b}_1)$, $K_2^A(\bar{b}_1) = K_2^B(\bar{b}_1)$, $K_3^A(\bar{b}_1) = K_3^B(\bar{b}_1)$, and the shear stress between fibres "1" and "2" is equal to τ_y; $\bar{\tau}_{1\text{-}2}(b_1) = \bar{\tau}_y$.

4. At $\xi = \infty$, stress concentration factors for all fibres are unity; $K_1^A(\infty) = 1$, $K_2^A(\infty) = 1$, $K_3^A(\infty) = 1$.

A_1 to A_6, B_1 to B_6, C_1 to C_6, $\bar{b}_1$ and $\bar{b}_2$ are independent of ξ but dependent on the stress level σ_f. As $1/\bar{\tau}_y$, which is re-defined as $\bar{\sigma}_f$, is given by $(\sigma_f/\tau_y)\,(G_m A_f/E_f d_{av} h)^{1/2}$ from Equation 13, it can be regarded as the non-dimensional stress level. In the calculation, giving various values of $\bar{\sigma}_f$, we can obtain numerically the above values for each $\bar{\sigma}_f$. Then stress concentration factors and shear stresses between each fibre can be calculated using Equations 45 to 48.

3. Examples of results of calculation by the present method

3.1. Stress concentration in intact fibres at $x = 0$ in Stage I

Some examples of results of calculation of stress concentration in "1" and "3" fibres at $x = 0$, $K_1(0)$ and $K_3(0)$, respectively, in Stage I are shown in Fig. 3. The following four cases (a) to (d) were taken as examples.

Case (a): f_1 varies under the fixed values of $f_2 = f_3 = 1$. As $f_1 + f_2 + f_3 + f_4$ is equal to 4 in definition, f_4 is given by $2 - f_1$.

Case (b): f_2 varies under the fixed values of $f_1 = f_4 = 1$. In this case, f_3 is given by $2 - f_2$.

Case (c): f_1 varies under the fixed values of $f_2 = 0.5$ and $f_3 = 1.5$. f_4 is given by $2 - f_1$.

Case (d): f_1 varies under the conditions of $f_1 = f_4$ and $f_2 = f_3$. f_2, f_3 and f_4 are given by $2 - f_1$, $2 - f_1$ and f_1, respectively. In this case the model composite shown in Fig. 2 becomes symmetric with the centre line of broken fibres. Therefore, $K_1(0)$ is equal to $K_3(0)$.

The effects of fibre spacing on $K_1(0)$ and $K_3(0)$ in Stage I could be summarized as follows.

(i) The larger the non-uniformity of fibre spacing, the larger the deviation of stress concentrations in the intact fibres from those for uniform fibre spacing ($f_1 = f_2 = f_3 = f_4 = 1$), as known from Cases C (a), (b) and (d).

(ii) In all cases, the larger the n, the higher the $K_1(0)$ and $K_3(0)$.

(iii) The spacings between broken and intact fibres (f_2 and f_3) have a more predominant effect on stress concentrations in intact fibres than those between intact and the next fibres (f_1 and f_4), as known by comparing Cases (a), (b) and (d) with each other. The narrower the former spacings, the higher the stress concentrations in intact fibres.

(iv) When the spacings between broken and intact fibres are given, the wider the spacings between intact and the next fibres, the higher is the stress concentrations in the intact fibres, as known from Cases (a) and (c).

The variation of $K_1(0)$ and $K_3(0)$ as a function of stress level in Stages II and III will be calculated in Section 3.3 using the conditions shown by arrows in Fig. 3.

3.2. Shear stress concentration between broken and intact fibres at $x = 0$

With increasing stress level, the matrix, corresponding to a smaller value between f_2 and f_3, begins yielding in shear at first at $x = 0$. After further loading, the matrix, corresponding to larger value between f_2 and f_3, also yields. Some examples of variation of shear stress between "1" and "2" fibres, $\tau_{1\text{-}2}$ and that between "2" and "3" fibres $\tau_{2\text{-}3}$ at $x = 0$ are shown as a

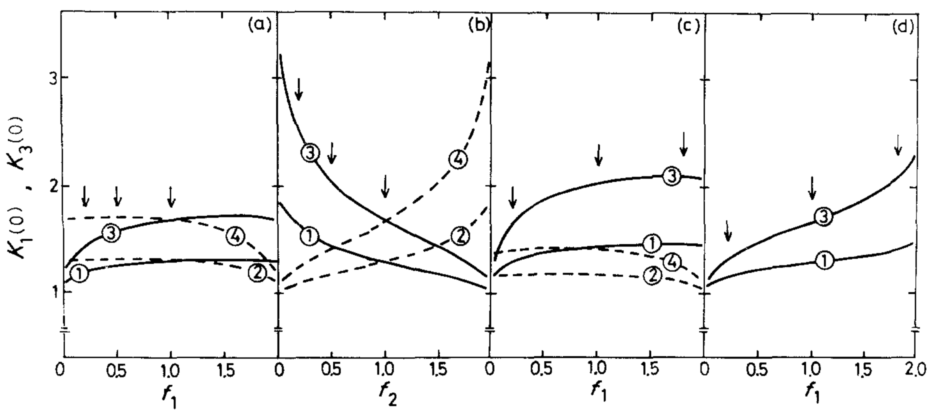

Figure 3 Variations of (1, 3) $K_1(0)$ and (2, 4) $K_3(0)$ in Stage I for Cases (a), (b), (c) and (d). (1, 2) $n = 1$, (3, 4) $n = 3$. The conditions shown by arrows will be used for calculation of $K_1(0)$ and $K_3(0)$ in Stages II and III in Section 3.3.

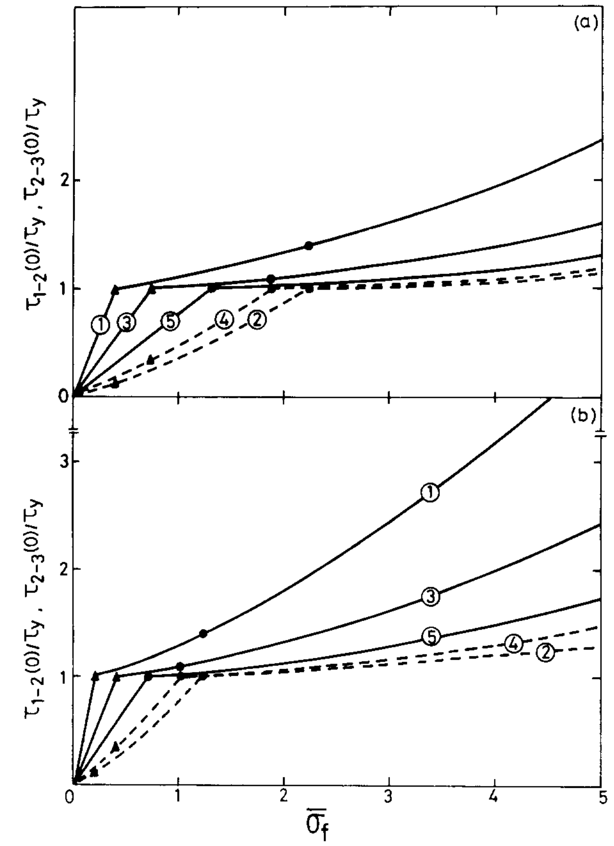

Figure 4 Variations of (1, 3, 5) $\tau_{1-2}(0)/\tau_y$ and (2, 4) $\tau_{2-3}(0)/\tau_y$ as a function of $\bar{\sigma}_f$ under the conditions of $\beta = 0.05$, $f_1 = f_4 = 1$, $f_2 = 0.2$, 0.5 and 1, and n = (a)1 and (b)3. (▲) and (●) Transition points at which Stages II and III arise, respectively. (1, 2) $f_1 = 1$, $f_2 = 0.2, f_3 = 1.8, f_4 = 1$; (3, 4) $f_1 = f_4 = 1, f_2 = 0.5, f_3 = 1.5$; (5) $f_1 = f_2 = f_3 = f_4 = 1$.

function of stress level, $\bar{\sigma}_f$, in Fig. 4, where the shear stress is normalized with respect to τ_y. In Fig. 4, ▲ and ● refer to the transition points from Stages I to II and from II to III, respectively. In these examples, f_2 was taken to be smaller than f_3. Therefore $\tau_{1-2}(0)/\tau_y$ became unity prior to $\tau_{2-3}(0)/\tau_y$. Below the stress level, corresponding to $\tau_{1-2}(0)/\tau_y = 1$ i.e. in Stage I, both $\tau_{1-2}(0)$ and $\tau_{2-3}(0)$ increase linearly with increasing stress level. Beyond this stress level, i.e. in Stage II, they increase non-linearly with increasing stress level until $\tau_{2-3}(0)/\tau_y$ becomes unity and the matrix between "2" and "3" begins to yield. After the yielding of this matrix, i.e. in Stage III, they also increase non-linearly. In the case of $f_1 = f_2 = f_3 = f_4 = 1$, $\tau_{1-2}(0)$ is equal to $\tau_{2-3}(0)$ due to the symmetry of the model composite, so that the "1"–"2" and "2"–"3" matrices surrounding the broken fibres yield simultaneously. Therefore Stage II does not exist in this case. Fig. 4 shows that (i) the larger the non-uniformity of the fibre, the larger the deviation of shear stress in the matrix from that for uniform spacing (curve 5) and (ii) the larger the n, the higher is the shear stress at the same stress level and therefore the earlier will the yielding of the matrix, begin.

3.3. Stress concentration in intact fibres at $x = 0$ as a function of stress level

The stress concentration factors in intact fibres at $x = 0$, $K_1(0)$ and $K_3(0)$, as a function of stress level

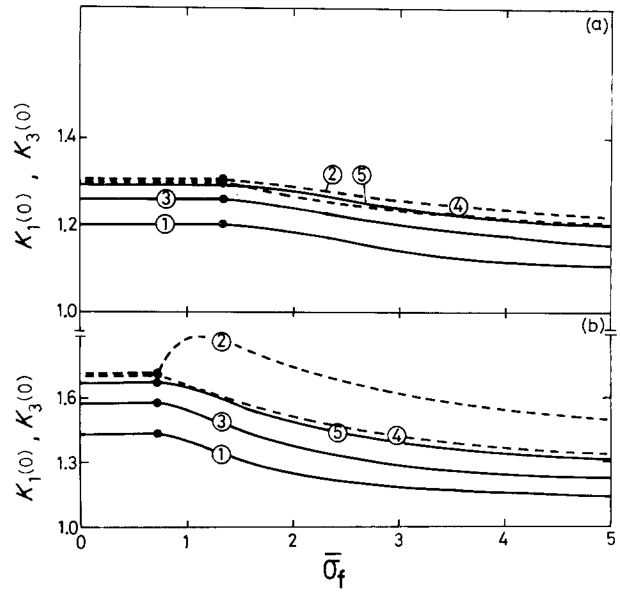

Figure 5 Variations of (1, 3, 5) $K_1(0)$ and (2, 4) $K_3(0)$ as a function of $\bar{\sigma}_f$ under conditions of $\beta = 0.05, f_2 = f_3 = 1, f_1 = 0.2$, 0.5 and 1, and n = (a)1 and (b)3. (1, 2) $f_1 = 0.2, f_2 = f_3 = 1, f_4 = 1.8$; (3, 4) $f_1 = 0.5, f_2 = f_3 = 1, f_4 = 1.5$; (5) $f_1 = f_2 = f_3 = f_4 = 1$.

are calculated for some examples for $\beta = 0.05$, as shown in Figs 5 to 8. The conditions in Figs 5 to 8 were taken from those shown by arrows in Figs 3a to d, respectively. The effect of non-uniformity of fibres spacing on $K_1(0)$ and $K_3(0)$ could be summarized as follows.

(i) The larger the non-uniformity of fibre spacing, the larger the deviation of $K_1(0)$ and $K_3(0)$ from those for uniform spacing.

(ii) The larger the difference between f_1 and f_4 for $f_2 = f_3 = 1$, and the larger the difference between f_2

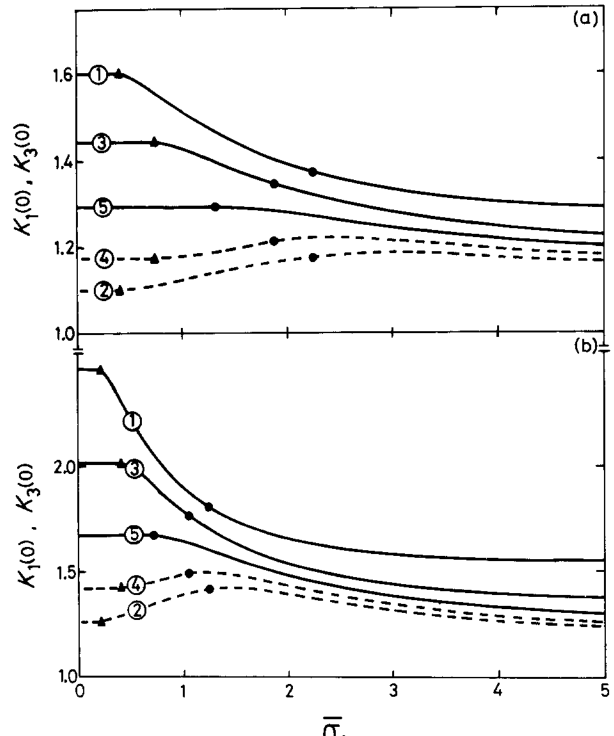

Figure 6 Variations of (1, 3, 5) $K_1(0)$ and (2, 4) $K_3(0)$ as a function of $\bar{\sigma}_f$ under conditions of $\beta = 0.05, f_1 = f_4 = 1, f_2 = 0.2$, 0.5 and 1, and n = (a)1 and (b)3. (1, 2) $f_1 = 1, f_2 = 0.2, f_3 = 1.8, f_4 = 1$; (3, 4) $f_1 = 1, f_2 = 0.5, f_3 = 1.5, f_4 = 1$; (5) $f_1 = f_2 = f_3 = f_4 = 1$.

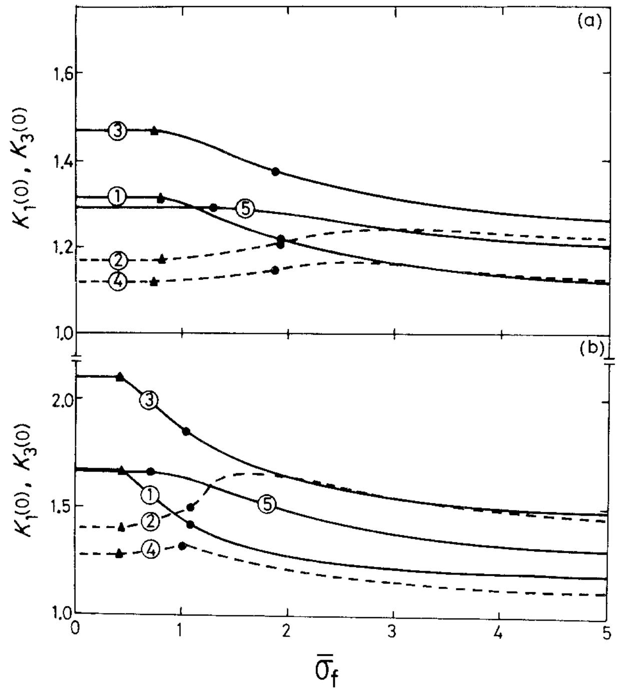

Figure 7 Variations of (1, 3, 5) $K_1(0)$ and (2, 4) $K_3(0)$ as a function of $\bar{\sigma}_f$ under conditions of $\beta = 0.05, f_1 = 0.2$ and $1.8, f_2 = 0.5$ and $f_3 = 1.5$, and $n =$ (a)1 and (b)3. The variations of $K_1(0)$ $(= K_3(0))$ under the condition of uniform fibre spacing, shown by curve 5, are also superimposed for comparison. (1, 2) $f_1 = 0.2$, $f_2 = 0.5$, $f_3 = 1.5, f_4 = 1.8$; (3, 4) $f_1 = 1.8, f_2 = 0.5, f_3 = 1.5, f_4 = 0.2$; (5) $f_1 = f_2 = f_3 = f_4 = 1$.

and f_3 for $f_1 = f_4 = 1$, the larger the difference between $K_1(0)$ and $K_3(0)$.

(iii) The larger the n, the higher the $K_1(0)$ and $K_3(0)$ at the same stress level.

(iv) The larger the n, Stages II and III arise at lower stress level.

(v) The variation of $K_1(0)$ remains constant in Stage I, but decreases in Stages II and III under the condition of $f_2 \leqslant f_3$. On the other hand, the variation of $K_3(0)$ is very complex; it sometimes increases in Stages II and III but sometimes decreases with increasing stress level. It seems to be important to point out that $K_3(0)$ can still increase after yielding of the matrices between "1" and "2" and between "2" and "3" fibres in Stage III, although the yielding of the matrix acts to reduce stress concentration in intact fibres when fibre spacing is uniform. Concerning the increment of $K_3(0)$ in Stage II and the initial region in Stage III, it was found from Figs 5, 6 and 7 that a greater increment in $K_3(0)$ occurs for larger values of n, for a greater difference between f_2 and f_3 when $f_1 = f_4$ and for a greater difference between f_1 and f_4 when $f_2 = f_3$.

(vi) Beyond the stress level where $K_3(0)$ increases, both $K_1(0)$ and $K_3(0)$ decrease monotonically with increasing stress level, and approach unity.

(vii) In the case of a symmetric array of fibres shown by curve 5 in Figs 5, 6 and 7, and shown by Curves 1, 2 and 3 in Fig. 8, the matrices between "1" and "2" fibres and that between "2" and "3" fibres yields simultaneously, according to which Stage II arises after Stage I. In this case $K_1(0)$ is equal to $K_3(0)$, and both $K_1(0)$ and $K_3(0)$ decrease with increasing stress level in Stage III.

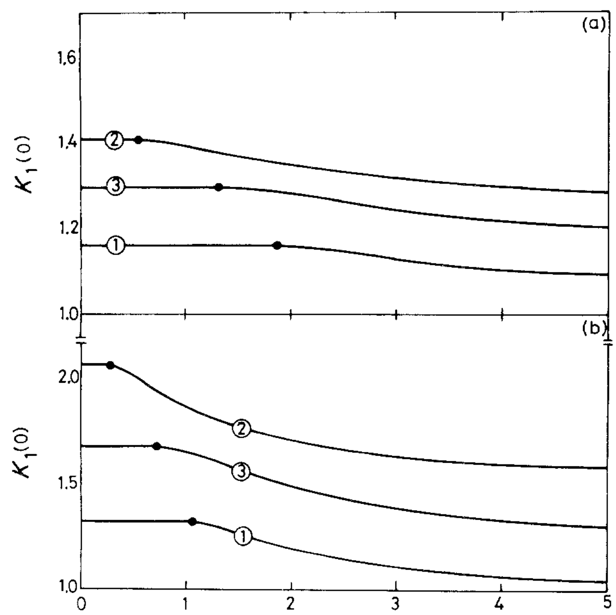

Figure 8 Variations of $K_1(0)$ $(= K_3(0))$ as a funciton of $\bar{\sigma}_f$ under conditions of $\beta = 0.05, f_1 = 0.2$, 1.0 and $1.8, f_1 = f_4$ and $f_2 = f_3$, and $n =$ (a)1 and (b)3. 1, $f_1 = 0.2, f_2 = 1.8, f_3 = 1.8, f_4 = 0.2$; 2, $f_1 = 1.8, f_2 = 0.2, f_3 = 0.2, f_4 = 1.8$; 3, $f_1 = f_2 = f_3 = f_4 = 1$.

3.4. Effects of strain hardening of matrix on stress concentration factors in intact fibres and shear stress concentration between broken and intact fibres at $x = 0$

An example of variations of $K_1(0)$, $K_3(0)$, $\tau_{1\text{-}2}(0)/\tau_y$ and $\tau_{2\text{-}3}(0)/\tau_y$ as a function of $\bar{\sigma}_f$ for $n = 3, f_1 = 1$,

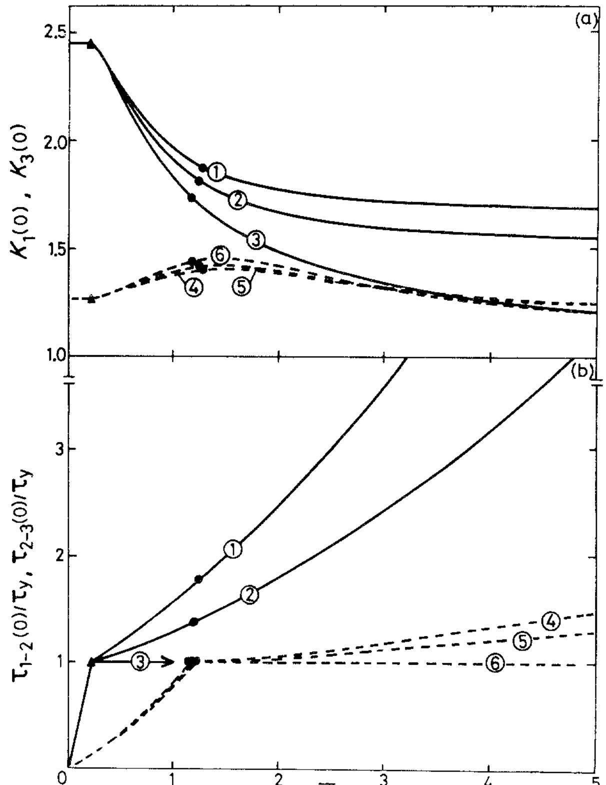

Figure 9 Influence of shear hardening of matrix, β, on (a) $K_1(0)$ (1–3) and $K_3(0)$ (4–6), and (b) $\tau_{1\text{-}2}(0)/\tau_y$ (1–3) and $\tau_{2\text{-}3}(0)/\tau_y$ (4–6), under conditions of $f_1 = 1$, $f_2 = 0.2$, $f_3 = 0.2$, $f_3 = 1.8$ and $f_4 = 1$, and $n = 1$. (1, 4) $\beta = 0.1$; (2, 5) $\beta = 0.05$; (3, 6) $\beta = 0$.

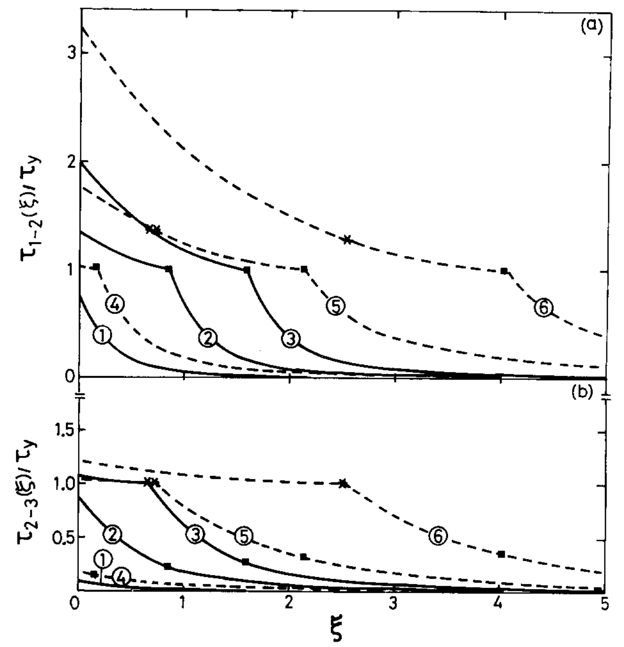

Figure 10 Variations of (a) $\tau_{1\text{-}2}(\xi)/\tau_y$ and (b) $\tau_{2\text{-}3}(\xi)/\tau_y$ as a function of ξ at the stress levels of $\bar{\sigma}_f$ = (1, 4) 0.3, (2, 5) 2 and (3, 6) 4, under conditions of $\beta = 0.05, f_1 = 1, f_2 = 0.2, f_3 = 1.8$ and $f_4 = 1$, and n = (1–3) 1 and (4–6) 3. (★) and (■) Transition points from Regions C to B and B to A, respectively.

$f_2 = 0.2, f_3 = 1.8, f_4 = 1$ and $\beta = 0$, 0.05 and 0.1 is shown in Fig. 9, which demonstrates the effects of the size of β. Not only the example shown in Fig. 9 but also other examples were calculated. From the calculation, the effects of strain hardening on stress concentrations could be summarized as follows. (Bear the case of $f_2 \leqslant f_3$ in mind.)

(i) The larger β, the more slowly decreases $K_1(0)$ with increasing $\bar{\sigma}_f$, and correspondingly the larger β, the more rapidly increases $\tau_{1\text{-}2}(0)/\tau_y$.

(ii) Under a fixed condition of non-uniformity of fibre spacing, $K_3(0)$ for small β can become higher than that for large β, while $K_1(0)$ for small β decreases more rapidly than that for large β with increasing stress level, as typically shown in Fig. 9.

(iii) When β becomes large, $\tau_{1\text{-}2}(0)/\tau_y$ becomes very large, especially under the condition where n is large and f_2 is much smaller than f_3.

3.5. Stress concentration in broken and intact fibres and shear stress concentration between broken and intact fibres in the fibre direction

Fig. 10 shows the shear stress distribution along the x-axis as a function of ξ, for examples of $n = 1$ and 3, $\beta = 0.05, f_1 = 1, f_2 = 0.2, f_3 = 1.8$ and $f_4 = 1$ at stress levels of $\bar{\sigma}_f = 0.3$, 2 and 4. ★ and ■ refer to the transition points at which Regions B and A arise, respectively. In the present examples, for $n = 1$, $\bar{\sigma}_f = 0.3$, 2 and 4 correspond to Stages I, II and III, respectively, for $n = 3$, $\bar{\sigma}_f = 0.3$ to Stage II and $\bar{\sigma}_f = 2$ and 4 to Stage III. Therefore, for $n = 1$, there is only Region A at $\bar{\sigma}_f = 0.3$, there are Regions A and B at $\bar{\sigma}_f = 2$ and Regions A, B and C at $\bar{\sigma}_f = 4$. For $n = 3$, there are Regions A and B at $\bar{\sigma}_f = 0.3$ and Regions A, B and C at $\bar{\sigma}_f = 2$ and 4. The $\tau_{1\text{-}2}(\xi)$ decreases slowly in Regions B and C but rapidly in Region A with increasing ξ while the $\tau_{2\text{-}3}(\xi)$ decreases slowly in Regions A and C but rapidly in Region B. The variation of stress concentration factors of $K_1(\xi)$, $K_2(\xi)$ and $K_3(\xi)$ under conditions the same as those shown in Fig. 10, is shown in Fig. 11. $K_1(\xi)$ and $K_3(\xi)$ decrease and $K_2(\xi)$ increase with increasing ξ, approaching unity at large ξ. The larger the n and the higher the applied stress level $\bar{\sigma}_f$, the larger becomes the distance of disturbances of stresses from $\xi = 0$.

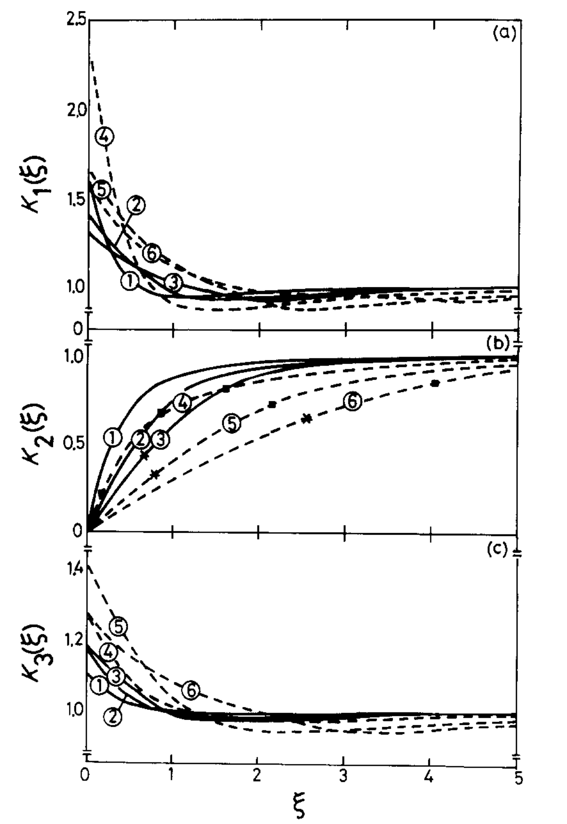

Figure 11 Variations of (a) $K_1(\xi)$, (b) $K_2(\xi)$ and (c) $K_3(\xi)$ as a function of ξ at the stress levels of $\bar{\sigma}_f$ = (1, 4) 0.3, (2, 5) 2 and (3, 6) 4, under conditions of $\beta = 0.05, f_1 = 1, f_2 = 0.2, f_3 = 1.8$ and $f_4 = 1$, and n = (1–3) 1 and (4–6) 3.

4. Conclusions

A method of calculation of stress concentration in intact fibres caused by broken fibres and shear stress concentration between broken and intact fibres in strain hardenable metal matrix composites in which the fibre spacing is not uniform has been presented. It was shown that the narrower the spacing between broken and intact fibres and the wider the spacing between intact and the next fibres, the higher become the tensile stress concentration in intact fibres and the shear stress concentrations between broken and intact fibres. When the number of broken fibres was large and the strain hardening of the matrix was high, this tendency was enhanced.

References

1. J. M. HEDGEPATH, NASA TN D-882 (1961).
2. W. B. FICHTER, NASA TN D-5453 (1969).
3. C. ZWEBEN, *Engng Fract. Mech.* **6** (1974) 1.
4. S. OCHIAI, K. ABE and K. OSAMURA, *Z. Metallkunde* **76** (1985) 299.
5. S. OCHIAI and K. OSAMURA, *J. Mater. Sci.* **24** (1989) 3536.

Received 31 August 1988
and accepted 12 January 1989